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Multiplicative perturbation bounds for spectral and singular value decompositions ☆

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Abstract

Let H be a Hermitian matrix, and $\tilde{H} = D^*HD$ be its perturbed matrix. In this paper, the multiplicative perturbations for both spectral decompositions and singular value decompositions are studied and some new perturbation bounds for these decompositions are presented. Our results improve some existing bounds.

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1. Introduction

Let H be an $n \times n$ matrix, and $\tilde{H}$ be its perturbed matrix. There are two different perturbation models:

- Additive perturbation $\tilde{H} = H + \Delta H$.
- Multiplicative perturbation $\tilde{H} = D^*HD$, where D is nonsingular and close to the identity matrix.

Some classical perturbation bounds for eigenvalues, eigenspaces, singular values and singular subspaces were given for the additive perturbation in general, e.g., Hoffman–Wielandt theorem for eigenvalues (see [3]) and the $\sin \Theta$ theorem for eigenspaces (see [2]), these error bounds are the best possible for arbitrary perturbation. Recently, some researchers have paid attention to the multiplicative perturbation because of applications (e.g., see [4–6] and references therein).

In this paper we focus on studying the multiplicative perturbation bound. Let H and $\tilde{H} = D^*HD$ be two $n \times n$ Hermitian matrices with the following spectral decompositions

$$H = (U_1 \quad U_2) \begin{pmatrix} \wedge_1 & 0 \\ 0 & \wedge_2 \end{pmatrix} \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} \quad \text{and} \quad \tilde{H} = (\tilde{U}_1 \quad \tilde{U}_2) \begin{pmatrix} \tilde{\wedge}_1 & 0 \\ 0 & \tilde{\wedge}_2 \end{pmatrix} \begin{pmatrix} \tilde{U}_1^* \\ \tilde{U}_2^* \end{pmatrix}, \tag{1.1}$$

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where $U = (U_1\ U_2)$ and $\tilde{U} = (\tilde{U}_1\ \tilde{U}_2)$ are unitary, U_1 and $\tilde{U}_1$ are $n \times r$, and

$$\wedge_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r), \quad \wedge_2 = \mathrm{diag}(\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n), \tag{1.2}$$

$$\tilde{\wedge}_1 = \mathrm{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_r), \quad \tilde{\wedge}_2 = \mathrm{diag}(\tilde{\lambda}_{r+1}, \tilde{\lambda}_{r+2}, \ldots, \tilde{\lambda}_n). \tag{1.3}$$

The relative gaps were given by [7]

$$\varrho_{ij} = \min_{\lambda \in \lambda(\wedge_i), \tilde{\lambda} \in \lambda(\tilde{\wedge}_j)} \frac{|\lambda - \tilde{\lambda}|}{\sqrt{|\tilde{\lambda}|^2 + |\lambda|^2}}, \quad i, \ \ j = 1, 2.$$

The canonical angle $\Theta(U_1, \tilde{U}_1)$ between the column spaces $\Re(U_1)$ of U_1 and $\Re(\tilde{U}_1)$ of $\tilde{U}_1$ is defined by $\Theta(U_1, \tilde{U}_1) = \arccos(U_1^* \tilde{U}_1 \tilde{U}_1{}^* U_1)^{1/2}$ (e.g., see [7]).

Let H and $\tilde{H} = D^* H D$ be two $n \times n$ Hermitian matrices with spectral decompositions (1.1)–(1.3), where D is nonsingular. Li [7] obtained a bound as follows: if $\varrho_{12} > 0$, then

$$\| \sin \Theta(U_1, \tilde{U}_1) \|_F \leqslant \frac{1}{\varrho_{12}} \sqrt{\|(I - D^{-1}) U_1\|_F^2 + \|(I - D^*) U_1\|_F^2}. \tag{1.4}$$

For the eigenvalue case, Li [5,6] obtained

$$\sum_{i=1}^{n} \frac{|\lambda_i - \tilde{\lambda}_{\tau(i)}|^2}{|\tilde{\lambda}_{\tau(i)}|^2 + |\lambda_i|^2} \leqslant \|I - D^{-1}\|_F^2 + \|I - D^*\|_F^2 \tag{1.5}$$

for some permutation τ of the set $\langle n \rangle \equiv \{1, 2, \ldots, n\}$.

A variation of (1.4) was given by Chen and Li [1] as follows

$$\| \sin \Theta(U_1, \tilde{U}_1) \|_F \leqslant \frac{1}{2} \sqrt{\frac{1}{\varrho_{12}^2} + \frac{1}{\varrho_{21}^2}} \sqrt{\|I - D^{-1}\|_F^2 + \|I - D^*\|_F^2} \tag{1.6}$$

for $\varrho_{ij} > 0$, $i, j = 1, 2$ and $i \neq j$.

The bound in (1.6) is independent with U_1. This could make the bound sometimes bigger than (1.4). However we can improve (1.4) in next section.

Li and Sun [8] presented a new perturbation bound for spectral decompositions for the additive perturbations. It implies both Hoffman–Wielandt theorem for eigenvalues and Davis and Kahans' $\sin \Theta$ theorem for eigenspaces into one. This bound can be extended to the singular value decompositions (SVDs), too. It is natural to ask: can we obtain a bound that implies both Li's bounds (1.4) and (1.5)? In this paper we will give an affirmative answer for this question by an approach similar to [8] for the multiplicative perturbation. Also we extend this bound to singular value cases.

The organization of this paper is as follows. In Section 2 we consider relative bounds for multiplicative perturbations of Hermitian matrices. A combined perturbation bound for eigenvalues and eigenspaces is given. Some new bounds for eigenspaces are also given in this section. Our bounds are always sharper than the corresponding ones in (1.4) and (1.6). In Section 3, we extend the results in Section 2 to the singular value and the singular subspace.

2. The bound for spectral decompositions

Let $\mathscr{C}^{m \times n}$ be the set of all $m \times n$ matrices. In this section we present a multiplicative perturbation bound for eigenspaces and a combined perturbation bounds for eigensystems.

The following lemma can be found in [7].

Lemma 2.1 (*Li [7]*). *Let $\Omega \in \mathscr{C}^{s\times s}$ and $\Gamma \in \mathscr{C}^{t\times t}$ be two Hermitian matrices, and let $E, F \in \mathscr{C}^{s\times t}$. If $\lambda(\Omega)\cap\lambda(\Gamma)=\emptyset$, then $\Omega X - X\Gamma = \Omega E + F\Gamma$ has a unique solution $X \in \mathscr{C}^{s\times t}$, and moreover,*

$$\|X\|_F \leqslant \sqrt{(\|E\|_F^2 + \|F\|_F^2)} \Big/ \varrho,$$

where $\varrho = \min_{\lambda\in\lambda(\Omega),\tilde{\lambda}\in\lambda(\Gamma)} |\lambda - \tilde{\lambda}| \Big/ \sqrt{|\tilde{\lambda}|^2 + |\lambda|^2}$.

The following lemma is useful to prove the main result. Its proof can be found in the proof of Theorem 6.1 [5].

Lemma 2.2. *Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$ and $\widetilde{\Lambda} = \mathrm{diag}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_r)$ satisfy $\Lambda X - X\widetilde{\Lambda} = \Lambda E + F\widetilde{\Lambda}$. Then there is a permutation τ of $\langle r\rangle$ such that*

$$\sigma_{\min}(X) \sum_{i=1}^{r} \frac{|\lambda_i - \tilde{\lambda}_{\tau(i)}|^2}{|\tilde{\lambda}_{\tau(i)}|^2 + |\lambda_i|^2} \leqslant \|E\|_F^2 + \|F\|_F^2.$$

The following bound (2.2) is the combined form for eigenvalues and eigenspaces.

Theorem 2.3. *Let H and $\tilde{H} = D^*HD$ be two $n\times n$ Hermitian matrices with spectral decompositions* (1.1)–(1.3), *where D is nonsingular. Then*

$$\varrho_{12}^2 \|\sin\Theta(U_1, \tilde{U}_1)\|_F^2 + \varrho_{11}^2 \|\cos\Theta(U_1, \tilde{U}_1)\|_F^2 \leqslant \|(I - D^{-1})U_1\|_F^2 + \|(I - D^*)U_1\|_F^2 \tag{2.1}$$

and there is a permutation τ of $\langle r\rangle$ such that

$$\varrho_{12}^2 \|\sin\Theta(U_1, \tilde{U}_1)\|_F^2 + (1 - \|\sin\Theta(U_1, \tilde{U}_1)\|_2^2) \sum_{i=1}^{r} \frac{|\lambda_i - \tilde{\lambda}_{\tau(i)}|^2}{|\tilde{\lambda}_{\tau(i)}|^2 + |\lambda_i|^2}$$
$$\leqslant \|(I - D^{-1})U_1\|_F^2 + \|(I - D^*)U_1\|_F^2. \tag{2.2}$$

Proof. Clearly we have

$$\Delta H = \tilde{H} - H = \tilde{H}(I - D^{-1}) + (D^* - I)H. \tag{2.3}$$

Left- and right-multiplying (2.3) by $\widetilde{U}^*$ and U_1, respectively, leads to

$$\tilde{\wedge}\tilde{U}^*U_1 - \tilde{U}^*U_1\wedge_1 = \tilde{\wedge}\tilde{U}^*(I - D^{-1})U_1 + \tilde{U}^*(D^* - I)U_1\wedge_1.$$

It can be written as a 2×2 block form as follows

$$\begin{pmatrix} \tilde{\wedge}_1\tilde{U}_1^*U_1 - \tilde{U}_1^*U_1\wedge_1 \\ \tilde{\wedge}_2\tilde{U}_2^*U_1 - \tilde{U}_2^*U_1\wedge_1 \end{pmatrix} = \begin{pmatrix} \tilde{\wedge}_1\tilde{U}_1^*(I - D^{-1})U_1 + \tilde{U}_1^*(D^* - I)U_1\wedge_1 \\ \tilde{\wedge}_2\tilde{U}_2^*(I - D^{-1})U_1 + \tilde{U}_2^*(D^* - I)U_1\wedge_1 \end{pmatrix},$$

and thus

$$\tilde{\wedge}_2\tilde{U}_2^*U_1 - \tilde{U}_2^*U_1\wedge_1 = \tilde{\wedge}_2\tilde{U}_2^*(I - D^{-1})U_1 + \tilde{U}_2^*(D^* - I)U_1\wedge_1 \tag{2.4}$$

and

$$\tilde{\wedge}_1\tilde{U}_1^*U_1 - \tilde{U}_1^*U_1\wedge_1 = \tilde{\wedge}_1\tilde{U}_1^*(I - D^{-1})U_1 + \tilde{U}_1^*(D^* - I)U_1\wedge_1. \tag{2.5}$$

Applying Lemma 2.1 to (2.4) and (2.5) yields

$$\varrho_{12}^2 \|\tilde{U}_2^*U_1\|_F^2 \leqslant \|\tilde{U}_2^*(I - D^{-1})U_1\|_F^2 + \|\tilde{U}_2^*(D^* - I)U_1\|_F^2 \tag{2.6}$$

and

$$\varrho_{11}^2 \|\tilde{U}_1^*U_1\|_F^2 \leqslant \|\tilde{U}_1^*(I - D^{-1})U_1\|_F^2 + \|\tilde{U}_1^*(D^* - I)U_1\|_F^2. \tag{2.7}$$

Notice that

$$\|\tilde{U}_1^*(I-D^{-1})U_1\|_F^2+\|\tilde{U}_2^*(I-D^{-1})U_1\|_F^2=\|(I-D^{-1})U_1\|_F^2 \tag{2.8}$$

and

$$\|\tilde{U}_1^*(D^*-I)U_1\|_F^2+\|\tilde{U}_2^*(D^*-I)U_1\|_F^2=\|(D^*-I)U_1\|_F^2. \tag{2.9}$$

Combining (2.4)–(2.9) we have

$$\varrho_{12}^2\|\tilde{U}_1^*U_2\|_F^2+\varrho_{11}^2\|\tilde{U}_1^*U_1\|_F^2\leqslant\|(I-D^{-1})U_1\|_F^2+\|(D^*-I)U_1\|_F^2. \tag{2.10}$$

It follows from [9] that

$$\|\sin\Theta(U_1,\tilde{U}_1)\|_F=\|U_1^*\tilde{U}_2\|_F=\|U_2^*\tilde{U}_1\|_F. \tag{2.11}$$

By the definition of $\cos\Theta$ we have

$$\|\cos\Theta(U_1,\tilde{U}_1)\|_F=\|U_1^*\tilde{U}_1\|_F. \tag{2.12}$$

Then the desired bound (2.1) follows from (2.10)–(2.12).

In order to prove (2.2), applying Lemma 2.2 to (2.5) yields

$$\sigma_{\min}^2(\tilde{U}_1^*U_1^*)\sum_{i=1}^{r}\frac{|\lambda_i-\tilde{\lambda}_{\tau(i)}|^2}{|\tilde{\lambda}_{\tau(i)}|^2+|\lambda_i|^2}\leqslant\|\tilde{U}_1^*(I-D^{-1})U_1\|_F^2+\|\tilde{U}_1^*(D^*-I)U_1\|_F^2. \tag{2.13}$$

By the C–S decomposition theorem (e.g., see [9]) we have

$$\sigma_{\min}^2(\tilde{U}_1^*U_1^*)=1-\|\sin\Theta(U_1,\tilde{U}_1)\|_2^2. \tag{2.14}$$

Then the bound (2.2) follows from (2.6), (2.13) and (2.14). □

By the C–S decomposition theorem [9] we have

$$\|\sin\Theta(U_1,\tilde{U}_1)\|_F^2=r-\|\cos\Theta(U_1,\tilde{U}_1)\|_F^2. \tag{2.15}$$

Applying (2.15) to the bound (2.1) gives the following corollary.

Corollary 2.4. *Under the same assumption as in Theorem* 2.3 *we have*

$$(\varrho_{12}^2-\varrho_{11}^2)\|\sin\Theta(U_1,\tilde{U}_1)\|_F^2+r\varrho_{11}^2\leqslant\|(I-D^{-1})U_1\|_F^2+\|(I-D^*)U_1\|_F^2. \tag{2.16}$$

Remark 2.1. Since

$$(\varrho_{12}^2-\varrho_{11}^2)\|\sin\Theta(U_1,\tilde{U}_1)\|_F^2+r\varrho_{11}^2\geqslant\varrho_{12}^2\|\sin\Theta(U_1,\tilde{U}_1)\|_F^2$$

and

$$\varrho_{12}^2\|\sin\Theta(U_1,\tilde{U}_1)\|_F^2+\varrho_{11}^2\|\cos\Theta(U_1,\tilde{U}_1)\|_F^2\geqslant\varrho_{12}^2\|\sin\Theta(U_1,\tilde{U}_1)\|_F^2,$$

the bounds in (2.1) and (2.16) are always sharper than the one in (1.4). If we take $U_1=U$ and $\tilde{U}_1=\tilde{U}$, then $\|\sin\Theta(U_1,\tilde{U}_1)\|_F^2=\|\sin\Theta(U_1,\tilde{U}_1)\|_2^2=0$ and hence the bound (2.2) reduces to (1.5), which implies that our combined perturbation bound (2.2) for eigenvalues and eigenspaces implies the bounds (1.4) and (1.5).

By the bounds (2.2), (1.4) and (1.5) we can obtain the asymptotic bound below

$$\begin{aligned}&\varrho_{12}^2\|\sin\Theta(U_1,\tilde{U}_1)\|_F^2+\sum_{i=1}^{r}\frac{|\lambda_i-\tilde{\lambda}_{\tau(i)}|^2}{|\tilde{\lambda}_{\tau(i)}|^2+|\lambda_i|^2}\\&\quad\leqslant\|(I-D^{-1})U_1\|_F^2+\|(I-D^*)U_1\|_F^2+O((\|(I-D^{-1})U_1\|_F^2+\|(I-D^*)U_1\|_F^2)^2)\end{aligned} \tag{2.17}$$

for some permutation τ of $\langle n\rangle$.

The following theorem is a variation of (2.16).

Theorem 2.5. *Under the same assumption as in Theorem* 2.3 *we have*

$$(\varrho_{12}^2+\varrho_{21}^2-\varrho_{11}^2-\varrho_{22}^2)\|\sin\Theta(U_1,\tilde U_1)\|_F^2+r\varrho_{11}^2+(n-r)\varrho_{22}^2 \leqslant \|I-D^{-1}\|_F^2+\|I-D^*\|_F^2. \tag{2.18}$$

Proof. Left- and right-multiplying (2.3) by $\widetilde{U}^*$ and U_2, respectively, gives

$$\tilde\wedge\tilde U^*U_2-\tilde U^*U_1\wedge_2=\tilde\wedge\tilde U^*(I-D^{-1})U_2+\tilde U^*(D^*-I)U_1\wedge_2.$$

By a similar argument to the proof of Theorem 2.3 we have

$$\tilde\wedge_1\tilde U_1^*U_2-\tilde U_1^*U_2\wedge_2=\tilde\wedge_1\tilde U_1^*(I-D^{-1})U_2+\tilde U_1^*(D^*-I)U_2\wedge_2 \tag{2.19}$$

and

$$\tilde\wedge_2\tilde U_2^*U_2-\tilde U_2^*U_2\wedge_2=\tilde\wedge_2\tilde U_1^*(I-D^{-1})U_2+\tilde U_2^*(D^*-I)U_2\wedge_2. \tag{2.20}$$

Applying Lemma 2.1 to (2.19) and (2.20), respectively, we have

$$\varrho_{21}^2\|\tilde U_1^*U_2\|_F^2\leqslant\|\tilde U_1^*(I-D^{-1})U_2\|_F^2+\|\tilde U_1^*(D^*-I)U_2\|_F^2 \tag{2.21}$$

and

$$\varrho_{22}^2\|\tilde U_2^*U_2\|_F^2\leqslant\|\tilde U_2^*(I-D^{-1})U_2\|_F^2+\|\tilde U_2^*(D^*-I)U_2\|_F^2. \tag{2.22}$$

Notice that $\|\tilde U_1^*U_2\|_F^2=\|\tilde U_2^*U_1\|_F^2$,

$$\|\tilde U_1^*(I-D^{-1})U_2\|_F^2+\|\tilde U_1^*(I-D^{-1})U_1\|_F^2+\|\tilde U_2^*(I-D^{-1})U_1\|_F^2+\|\tilde U_2^*(I-D^{-1})U_2\|_F^2 = \|(I-D^{-1})\|_F^2$$

and

$$\|\tilde U_1^*(D^*-I)U_2\|_F^2+\|\tilde U_1^*(D^*-I)U_1\|_F^2+\|\tilde U_2^*(D^*-I)U_1\|_F^2+\|\tilde U_2^*(D^*-I)U_2\|_F^2 = \|(D^*-I)\|_F^2,$$

which together with (2.6), (2.7), (2.21) and (2.22) gives

$$(\varrho_{12}^2+\varrho_{12}^2)\|\tilde U_1^*U_2\|_F^2+\varrho_{11}^2\|\tilde U_1^*U_1\|_F^2+\varrho_{22}^2\|\tilde U_2^*U_2\|_F^2 \leqslant\|(I-D^{-1})\|_F^2+\|(D^*-I)\|_F^2. \tag{2.23}$$

Clearly,

$$r=\|U_1\|_F^2=\|\tilde U^*U_1\|_F^2=\|\tilde U_1^*U_1\|_F^2+\|\tilde U_2^*U_1\|_F^2.$$

Hence $\|\tilde U_1^*U_1\|_F^2=r-\|\tilde U_2^*U_1\|_F^2$. Similarly, we have $\|\tilde U_2^*U_2\|_F^2=n-r-\|\tilde U_2^*U_1\|_F^2$, which together with (2.23) gives the desired bound (2.18). □

Notice that $\|\sin\Theta(U_1,\tilde U_1)\|_F^2\leqslant\min\{r,n-r\}$. Hence by Theorem 2.5 it is easy to prove the following corollary.

Corollary 2.6. *Under the same assumption as in Theorem* 2.3 we have

$$\|\sin\Theta(U_1,\tilde U_1)\|_F\leqslant\frac{1}{\sqrt{\varrho_{12}^2+\varrho_{21}^2}}\sqrt{\|I-D^{-1}\|_F^2+\|I-D^*\|_F^2-\min\{\varrho_{11}^2,\varrho_{22}^2\}|n-2r|} \leqslant\frac{1}{\sqrt{\varrho_{12}^2+\varrho_{21}^2}}\sqrt{\|I-D^{-1}\|_F^2+\|I-D^*\|_F^2}. \tag{2.24}$$

Remark 2.2. By Corollary 2.6 it is easy to see that the bound in (2.24) is always sharper than the one in (1.6).

Remark 2.3. In order to compare Li's bound (1.4) with the bound (2.24), we take an example as follows.

Let

$$H = \begin{pmatrix} I & 0 \\ 0 & 2I \end{pmatrix}, \quad \widetilde{H} = (U_1, U_2) H \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix},$$

where $U = (U_1, U_2)$ is a unitary matrix, U_1 is $n \times r$. Then $\varrho_{12} = \varrho_{21}$. By Li's bound we have

$$\|\sin \Theta(I_1, U_1)\|_F \leqslant \frac{\sqrt{2}\|I_1 - U_1\|_F}{\varrho_{12}},$$

where $I_1 = (I, 0)^{\mathrm{T}}$ and $I_2 = (0, I)^{\mathrm{T}}$. By (2.24) we have

$$\|\sin \Theta(I_1, U_1)\|_F \leqslant \frac{\sqrt{2}\|I - U\|_F}{\sqrt{\varrho_{12}^2 + \varrho_{21}^2}} = \frac{\sqrt{\|I_1 - U_1\|_F^2 + \|I_2 - U_2\|_F^2}}{\varrho_{12}}. \tag{2.25}$$

If we take U with

$$\|I_2 - U_2\|_F < \|I_1 - U_1\|_F,$$

then

$$\frac{\sqrt{\|I_1 - U_1\|_F^2 + \|I_2 - U_2\|_F^2}}{\varrho_{12}} < \frac{\sqrt{2}\|I_1 - U_1\|_F}{\varrho_{12}},$$

which shows that the bound (2.24) is sharper than Li's bound (1.4) by (2.25). However, if we take

$$\|I_1 - U_1\|_F < \|I_2 - U_2\|_F,$$

it is easy to see that Li's bound (1.4) is better than the one in (2.24), which illustrates that neither of the bounds (1.4) and (2.24) is better than other.

3. The bound for SVDs

Let $A, \tilde{A} \in \mathscr{C}^{n \times n}$ have the SVDs:

$$A = U\Sigma V^* = (U_1 \quad U_2) \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} \tag{3.1}$$

and

$$\tilde{A} = \tilde{U}\tilde{\Sigma}\tilde{V}^* = (\tilde{U}_1 \quad \tilde{U}_2) \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^* \\ \tilde{V}_2^* \end{pmatrix}, \tag{3.2}$$

where $U = (\overset{r}{U_1} \quad \overset{n-r}{U_2})$, $\tilde{U} = (\overset{r}{\tilde{U}_1} \quad \overset{n-r}{\tilde{U}_2})$, $V = (\overset{r}{V_1} \quad \overset{n-r}{V_2})$ and $\tilde{V} = (\overset{r}{\tilde{V}_1} \quad \overset{n-r}{\tilde{V}_2})$ are $n \times n$ unitary, and

$$\Sigma_1 = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \quad \Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n), \tag{3.3}$$

$$\tilde{\Sigma}_1 = \operatorname{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_r), \quad \tilde{\Sigma}_2 = \operatorname{diag}(\tilde{\sigma}_{r+1}, \tilde{\sigma}_{r+2}, \ldots, \tilde{\sigma}_n). \tag{3.4}$$

The classical $\sin\theta$ theorem for singular subspaces for additive perturbation was given by Wedin [10]. For the multiplicity perturbation, i.e., $\tilde{A} = D_L^* A D_R$, Li [7] obtained the following bound

$$\begin{aligned} &\|\sin \Theta(U_1, \tilde{U}_1)\|_F^2 + \|\sin \Theta(V_1, \tilde{V}_1)\|_F^2 \\ &\quad \leqslant \frac{\|(I - D_L^{-1})U_1\|_F^2 + \|(I - D_R^{-1})V_1\|_F^2 + \|(I - D_L^*)U_1\|_F^2 + \|(I - D_R^*)V_1\|_F^2}{\eta_{12}^2}, \end{aligned} \tag{3.5}$$

where

$$\eta_{12} = \min_{\sigma\in\sigma(\Sigma_1),\tilde\sigma\in\sigma(\tilde\Sigma_2)} \frac{|\sigma-\tilde\sigma|}{\sqrt{|\tilde\sigma|^2+|\sigma|^2}}.$$

In this section some perturbation bounds in Section 2 will be extended for SVDs. Hence we consider the Jordan–Wielandt matrices

$$\mathbf{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & \tilde A^* \\ \tilde A & 0 \end{pmatrix}. \tag{3.6}$$

Let

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} V_1 & V_1 & V_2 & V_2 \\ U_1 & -U_1 & U_2 & -U_2 \end{pmatrix} \quad \text{and} \quad \tilde X = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde V_1 & \tilde V_1 & \tilde V_2 & \tilde V_2 \\ \tilde U_1 & -\tilde U_1 & \tilde U_2 & -\tilde U_2 \end{pmatrix}. \tag{3.7}$$

Then the spectral decompositions of $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = X\begin{pmatrix} \Sigma_1 & & & \\ & -\Sigma_1 & & \\ & & \Sigma_2 & \\ & & & -\Sigma_2 \end{pmatrix}X^* \quad \text{and} \quad \mathbf{B} = \tilde X\begin{pmatrix} \tilde\Sigma_1 & & & \\ & -\tilde\Sigma_1 & & \\ & & \tilde\Sigma_2 & \\ & & & -\tilde\Sigma_2 \end{pmatrix}\tilde X^*,$$

respectively.

Let

$$X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} V_1 & V_1 \\ U_1 & -U_1 \end{pmatrix} \quad \text{and} \quad \tilde X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde V_1 & \tilde V_1 \\ \tilde U_1 & -\tilde U_1 \end{pmatrix}. \tag{3.8}$$

Now we consider multiplicative perturbations for the perturbed matrix $\tilde A = D_L^* A D_R$, where D_L and D_R are nonsingular and close to the identity matrices. Let $\mathbf{A}$, $\mathbf{B}$, X, $\tilde X$, X_1 and $\tilde X_1$ as in (3.6)–(3.8). Then

$$\tilde{\mathbf{A}} = \begin{pmatrix} D_R^* & 0 \\ 0 & D_L^* \end{pmatrix}\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}\begin{pmatrix} D_R & 0 \\ 0 & D_L \end{pmatrix} \equiv D^*\mathbf{A}D.$$

Let

$$\eta_{ij} = \min_{\sigma\in\sigma(\Sigma_i),\tilde\sigma\in\sigma(\tilde\Sigma_j)} \frac{|\sigma-\tilde\sigma|}{\sqrt{|\tilde\sigma|^2+|\sigma|^2}}, \quad i,\ j = 1, 2.$$

Applying Theorem 2.3 to the matrices $\mathbf{A}$ and $\mathbf{B}$ we have the following theorems.

Theorem 3.1. *Let A and $\tilde A = D_L^* A D_R$ be two $n\times n$ nonsingular matrices with SVDs* (3.1)–(3.4). *Then*

$$\begin{aligned}(\eta_{12}^2-\eta_{11}^2)&(\|\sin\Theta(U_1,\tilde U_1)\|_F^2+\|\sin\Theta(V_1,\tilde V_1)\|_F^2)+2r\eta_{11}^2 \\ &\leqslant \|(I-D_L^{-1})U_1\|_F^2+\|(I-D_R^{-1})V_1\|_F^2+\|(I-D_L^*)U_1\|_F^2+\|(I-D_R^*)V_1\|_F^2\end{aligned} \tag{3.9}$$

and there is τ of $\langle r\rangle$ such that

$$\begin{aligned}&\eta_{12}^2(\|\sin\Theta(U_1,\tilde U_1)\|_F^2+\|\sin\Theta(V_1,\tilde V_1)\|_F^2) \\ &\quad + (2-\|\sin\Theta(U_1,\tilde U_1)\|_2^2-\|\sin\Theta(V_1,\tilde V_1)\|_2^2)\sum_{i=1}^{r}\frac{|\sigma_i-\tilde\sigma_{\tau(i)}|^2}{|\tilde\sigma_{\tau(i)}|^2+|\sigma_i|^2} \\ &\quad\leqslant \|(I-D_L^{-1})U_1\|_F^2+\|(I-D_R^{-1})V_1\|_F^2+\|(I-D_L^*)U_1\|_F^2+\|(I-D_R^*)V_1\|_F^2.\end{aligned} \tag{3.10}$$

Theorem 3.2. *Let A and $\tilde A = D_L^* A D_R$ be two $n\times n$ nonsingular matrices with SVDs* (3.1)–(3.4). *Then*

$$\begin{aligned}(\eta_{12}^2+\eta_{21}^2-\eta_{11}^2-\eta_{22}^2)&(\|\sin\Theta(U_1,\tilde U_1)\|_F^2+\|\sin\Theta(V_1,\tilde V_1)\|_F^2)+2r\eta_{11}^2+2(n-r)\eta_{22}^2 \\ &\leqslant \|I-D_L^{-1}\|_F^2+\|I-D_R^{-1}\|_F^2+\|I-D_L^*\|_F^2+\|I-D_R^*\|_F^2.\end{aligned} \tag{3.11}$$

Remark 3.3. By Theorem 3.1 it is easy to see that the bounds in (3.9) and (3.10) are sharper than the one in (3.5). As the same argument as in Remark 2.1, the bound in (3.10) is a combined perturbation bound for singular values and singular subspaces. When we take $U_1 = U$, $\tilde{U}_1 = \tilde{U}$, $V_1 = V$ and $\tilde{V}_1 = \tilde{V}$, then $\|\sin\Theta(U_1, \tilde{U}_1)\|_F^2 = \|\sin\Theta(V_1, \tilde{V}_1)\|_F^2 = 0$, and hence the bound (3.10) reduces to the following perturbation bound for singular values:

$$\sum_{i=1}^{n} \frac{|\sigma_i - \tilde{\sigma}_{\tau(i)}|^2}{|\tilde{\sigma}_{\tau(i)}|^2 + |\sigma_i|^2} \leqslant \frac{1}{2}(\|I - D_L^{-1}\|_F^2 + \|I - D_R^{-1}\|_F^2 + \|I - D_L^*\|_F^2 + \|I - D_R^*\|_F^2), \tag{3.12}$$

which illustrates that the bound (3.10) implies the bounds (3.5) and (3.12). It is noted that Li [5] obtained the following bound:

$$\sum_{i=1}^{n} \frac{|\sigma_i - \tilde{\sigma}_{\tau(i)}|^2}{|\tilde{\sigma}_{\tau(i)}|^2 + |\sigma_i|^2} \leqslant \frac{1}{8}(\|D_L^* - D_L^{-1}\|_F + \|D_R^* - D_R^{-1}\|_F)^2. \tag{3.13}$$

However, the bound in (3.12) is not better than the one in (3.13).

A simpler form of the bound (3.11) is given below:

$$\begin{aligned}&\|\sin\Theta(U_1, \tilde{U}_1)\|_F^2 + \|\sin\Theta(V_1, \tilde{V}_1)\|_F^2 \\ &\quad\leqslant \frac{\|I - D_L^{-1}\|_F^2 + \|I - D_R^{-1}\|_F^2 + \|I - D_L^*\|_F^2 + \|I - D_R^*\|_F^2}{\eta_{12}^2 + \eta_{21}^2}.\end{aligned} \tag{3.14}$$

The bound (3.14) is slightly different from the bound (3.5), which depends on two relative gaps η_{12} and η_{21}. However, neither of these bounds in (3.5) and (3.14) is better than the other. The following example illustrates that our bound is better.

Let

$$A = \begin{pmatrix} I & 0 \\ 0 & 2I \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} U_{11} & 0 \\ 0 & I \end{pmatrix} A \begin{pmatrix} V_{11}^* & 0 \\ 0 & I \end{pmatrix},$$

where U_{11} and V_{11} are unitary matrices of order r. Then $\eta_{12} = \eta_{21}$. By Li's bound we have

$$\|\sin\Theta(I_1, U_1)\|_F \leqslant \frac{1}{\eta_{12}}\sqrt{2(\|I - U_{11}\|_F^2 + \|I - V_{11}\|_F^2)}.$$

By (2.24) we have

$$\begin{aligned}\|\sin\Theta(I_1, U_1)\|_F &\leqslant \frac{1}{\sqrt{\eta_{12}^2 + \eta_{21}^2}}\sqrt{2(\|I - U_{11}\|_F^2 + \|I - V_{11}\|_F^2)} \\ &= \frac{1}{\eta_{12}}\sqrt{\|I - U_{11}\|_F^2 + \|I - V_{11}\|_F^2} \\ &< \frac{1}{\eta_{12}}\sqrt{2(\|I - U_{11}\|_F^2 + \|I - V_{11}\|_F^2)}.\end{aligned}$$

As the argument as in Remark 2.1, the following asymptotic bound follows from (3.10), (3.12) and (3.5):

$$\eta_{12}^2(\|\sin\Theta(U_1, \tilde{U}_1)\|_F^2 + \|\sin\Theta(V_1, \tilde{V}_1)\|_F^2) + 2\sum_{i=1}^{r} \frac{|\sigma_i - \tilde{\sigma}_{\tau(i)}|^2}{|\tilde{\sigma}_{\tau(i)}|^2 + |\sigma_i|^2} \leqslant l_D + \mathrm{O}(l_D^2),$$

where $l_D = \|(I - D_L^{-1})U_1\|_F^2 + \|(I - D_R^{-1})V_1\|_F^2 + \|(I - D_L^*)U_1\|_F^2 + \|(I - D_R^*)V_1\|_F^2$.

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